

Lovelock-Brans-Dicke gravity

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According to Lovelock's theorem, the Hilbert-Einstein and the Lovelock actions are indistinguishable from their field equations. However, they have different scalar-tensor counterparts, which correspond to the Brans-Dicke and the *Lovelock-Brans-Dicke* (LBD) gravities, respectively. In this paper the LBD model of alternative gravity with the Lagrangian density $\mathcal{L}_{\text{LBD}} = \frac{1}{16\pi} \left[\phi \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) - \frac{\omega}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right]$ is developed, where *RR and \mathcal{G} respectively denote the topological Chern-Pontryagin and Gauss-Bonnet invariants. The field equation, the kinematical and dynamical wave equations, and the constraint from energy-momentum conservation are all derived. It is shown that, the LBD gravity reduces to general relativity in the limit $\omega_L \rightarrow \infty$ unless the “topological balance condition” holds, and in vacuum it can be conformally transformed into the dynamical Chern-Simons gravity and the generalized Gauss-Bonnet dark energy with Horndeski-like or Galileon-like kinetics. Moreover, the LBD gravity allows for the late-time cosmic acceleration without dark energy. Finally, the LBD gravity is generalized into the Lovelock-scalar-tensor gravity, and its equivalence to fourth-order modified gravities is established. It is also emphasized that the standard expressions for the contributions of generalized Gauss-Bonnet dependence can be further simplified.

Key words: Lovelock's theorem, topological effects, modified gravity

I. INTRODUCTION

As an alternative to the various models of dark energy with large negative pressure that violates the standard energy conditions, the accelerated expansion of the Universe has inspired the reconsideration of relativistic gravity and modifications of general relativity (GR), which can explain the cosmic acceleration and reconstruct the entire expansion history without dark energy.

Such alternative and modified gravities actually encode the possible ways to go beyond Lovelock's theorem and its necessary conditions [1], which limit the second-order field equation in four dimensions to $R_{\mu\nu} - Rg_{\mu\nu}/2 + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}^{(m)}$, i.e. Einstein's equation supplemented by the cosmological constant Λ . These directions can allow for, for example, fourth and even higher order gravitational field equations [2–5], more than four spacetime dimensions [6, 7], extensions of pure pseudo-Riemannian geometry and metric gravity [7, 8], extra physical degrees of freedom [9–12], and nonminimal curvature-matter couplings [13, 14]. From a variational approach, these violations manifest themselves as different modifications of the Hilbert-Einstein action, such as extra curvature invariants, scalar fields, and non-Riemannian geometric variables.

For the Lovelock action in Lovelock's theorem and the Hilbert-Einstein- Λ action, it is well known that they yield the same field equation and thus are indistinguishable by their gravitational effects. When reconsidering Lovelock's theorem, we cannot help but ask whether the effects of these two actions are really the same in all possible aspects. Is there any way for the two topological sources in the Lovelock action to show nontrivial consequences? As a possible answer to this

question, we propose the Lovelock-Brans-Dicke gravity.

This paper is organized as follows. In Sec. II, the Lovelock-Brans-Dicke gravity is introduced based on Lovelock's theorem, and its gravitational and wave equations are derived in Sec. III. Section IV studies the behaviors at the infinite-Lovelock-parameter limit $\omega_L \rightarrow \infty$, and Sec. V derives the constraint from energy-momentum conservation. Section VI shows that in vacuum the Lovelock-Brans-Dicke gravity can be conformally transformed into the dynamical Chern-Simons gravity and the generalized Gauss-Bonnet dark energy with Horndeski-like or Galileon-like kinetics. Then the possibility of realizing the acceleration phase for the late-time Universe is discussed in Sec. VII. Finally, in Sec. VIII the Lovelock-Brans-Dicke theory is extended to the Lovelock-scalar-tensor gravity, and its equivalence to fourth-order modified gravities is analyzed. Throughout this paper, we adopt the sign conventions $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$, $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} + \dots$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ with the metric signature $(-, +, +, +)$.

II. LOVELOCK-BRANS-DICKE ACTION

An algebraic Riemannian invariant $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(g_{\alpha\beta}, R_{\alpha\mu\beta\nu})$ in the action $\int d^4x \sqrt{-g} \tilde{\mathcal{R}}$ generally leads to fourth-order gravitational field equations by the variational derivative

$$\frac{\delta(\sqrt{-g}\tilde{\mathcal{R}})}{\delta g^{\mu\nu}} = \frac{\partial(\sqrt{-g}\tilde{\mathcal{R}})}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial(\sqrt{-g}\tilde{\mathcal{R}})}{\partial(\partial_\alpha g^{\mu\nu})} + \partial_\alpha \partial_\beta \frac{\partial(\sqrt{-g}\tilde{\mathcal{R}})}{\partial(\partial_\alpha \partial_\beta g^{\mu\nu})}. \quad (1)$$

Lovelock found out that in *four* dimensions the most general action leading to second-order field equations is [1]

$$S = \int d^4x \sqrt{-g} \mathcal{L} + S_m \quad \text{with} \quad (2)$$

$$\mathcal{L} = \frac{1}{16\pi G} \left(R - 2\Lambda + \frac{a}{2\sqrt{-g}} \epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}{}_{\gamma\delta} R^{\alpha\beta\gamma\delta} + b\mathcal{G} \right),$$

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where Λ is the cosmological constant, $\{a, b\}$ are dimensional coupling constants, and without any loss of generality we have set the coefficient of R equal to one. Also, $\epsilon_{\alpha\beta\mu\nu}$ refers to the totally antisymmetric Levi-Civita pseudotensor with $\epsilon_{0123} = \sqrt{-g}$, $\epsilon^{0123} = \frac{1}{\sqrt{-g}}$, and $\{\epsilon_{\alpha\beta\mu\nu}, \epsilon^{\alpha\beta\mu\nu}\}$ can be obtained from each other by raising or lowering the indices with the metric tensor. In Eq.(2), $\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta}$ and \mathcal{G} respectively refer to the Chern-Pontryagin density and the Gauss-Bonnet invariant, with

$$\mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}. \quad (3)$$

The variational derivatives $\delta(\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta})/\delta g^{\mu\nu}$ and $\delta(\sqrt{-g}\mathcal{G})/\delta g^{\mu\nu}$ yield total derivatives which serve as boundary terms in varying the full action Eq.(2). The Chern-Pontryagin scalar $\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta}$ is proportional to the divergence of the topological Chern-Simons four-current K^μ [11],

$$\begin{aligned} \epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta} &= -8 \partial_\mu K^\mu \quad \text{with} \\ K^\mu &= \epsilon^{\mu\alpha\beta\gamma} \left(\frac{1}{2} \Gamma_{\alpha\tau}^\xi \partial_\beta \Gamma_{\gamma\xi}^\tau + \frac{1}{3} \Gamma_{\alpha\tau}^\xi \Gamma_{\beta\eta}^\tau \Gamma_{\gamma\xi}^\eta \right), \end{aligned} \quad (4)$$

and similarly, the topological current for the Gauss-Bonnet invariant is (see Refs.[15, 16] for earlier discussion and Ref.[17] for further clarification)

$$\begin{aligned} \sqrt{-g}\mathcal{G} &= -\partial_\mu J^\mu \quad \text{with} \\ J^\mu &= \sqrt{-g} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{\rho\sigma}^{\xi\zeta} \Gamma_{\xi\alpha}^\rho \left(\frac{1}{2} R_{\zeta\beta\gamma}^\sigma - \frac{1}{3} \Gamma_{\lambda\beta}^\sigma \Gamma_{\zeta\gamma}^\lambda \right). \end{aligned} \quad (5)$$

Hence, the covariant densities $\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta}$ and $\sqrt{-g}\mathcal{G}$ in Eq.(2) make no contribution to the field equation $\delta S/\delta g^{\mu\nu} = 0$.

According to Lovelock's theorem, one cannot tell whether Einstein's equation $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}^{(m)}$ comes from the customary Hilbert-Einstein action

$$S_{\text{HE}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_m, \quad (6)$$

or from the induced Lovelock action¹

$$\begin{aligned} S_L &= \int d^4x \sqrt{-g} \mathcal{L}_L + S_m \quad \text{with} \\ \mathcal{L}_L &= \frac{1}{16\pi G} \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right), \end{aligned} \quad (7)$$

where for simplicity we switch to the denotation

$${}^*RR := \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} R^{\alpha\beta\gamma\delta} \quad (8)$$

for the Chern-Pontryagin density, as the symbol *RR has been widely used in the literature of Chern-Simons gravity

[11, 18, 19]. In Eqs.(2), (6) and (7), the matter action S_m is given in terms of the matter Lagrangian density \mathcal{L}_m by $S_m = \int d^4x \sqrt{-g} \mathcal{L}_m$, and the stress-energy-momentum density tensor $T_{\mu\nu}^{(m)}$ is defined in the usual way by [20]

$$\begin{aligned} \delta S_m &= -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu}^{(m)} \delta g^{\mu\nu} \quad \text{with} \\ T_{\mu\nu}^{(m)} &:= \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}. \end{aligned} \quad (9)$$

The indistinguishability between S_L and S_{HE} from their field equations begs the question: Does Einstein's equation come from S_L or S_{HE} ? Is there any way to discriminate them?

Recall that GR from S_{HE} has a fundamental scalar-tensor counterpart, the Brans-Dicke gravity [9],

$$\begin{aligned} S_{\text{BD}} &= \int d^4x \sqrt{-g} \mathcal{L}_{\text{BD}} + S_m \quad \text{with} \\ \mathcal{L}_{\text{BD}} &= \frac{1}{16\pi} \left(\phi R - \frac{\omega_{\text{BD}}}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right), \end{aligned} \quad (10)$$

which proves to be a successful alternative to GR that passes all typical GR tests [21], and it is related to GR by

$$\begin{aligned} \mathcal{L}_{\text{HE}} &= \frac{R}{16\pi G} \\ \Rightarrow \mathcal{L}_{\text{BD}} &= \frac{1}{16\pi} \left(\phi R - \frac{\omega_{\text{BD}}}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right). \end{aligned} \quad (11)$$

That is to say, Brans-Dicke firstly replaces the matter-gravity coupling constant G with a pointwise scalar field $\phi(x^\alpha)$ in accordance with the spirit of Mach's principle, $G \mapsto \phi^{-1}$, and further adds to the action a formally canonical kinetic term $-\frac{\omega_{\text{BD}}}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi$ governing the kinetics of $\phi(x^\alpha)$. Applying this prescription to the Lovelock action Eq.(7), we obtain

$$\begin{aligned} \mathcal{L}_L &= \frac{1}{16\pi G} \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) \\ \Rightarrow \mathcal{L}_{\text{LBD}} &= \frac{1}{16\pi} \left[\phi \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) - \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right], \end{aligned} \quad (12)$$

where the Lovelock parameter ω_L is a dimensionless constant. Based on Eq.(12), we obtain what we dub as the *Lovelock-Brans-Dicke* (henceforth LBD) gravity with the action

$$S_{\text{LBD}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{LBD}} + S_m, \quad (13)$$

or the *Lanczos-Lovelock-Brans-Dicke gravity*, as Lovelock's theorem is based on Lanczos' discovery that an isolated *RR or \mathcal{G} in the action does not affect the field equation [22].

Unlike the $\delta({}^*RR)/\delta g^{\mu\nu}$ and $\delta(\sqrt{-g}\mathcal{G})/\delta g^{\mu\nu}$ in $\delta S_L/\delta g^{\mu\nu}$, the $[\phi\delta({}^*RR)]/\delta g^{\mu\nu}$ and $[\phi\delta(\sqrt{-g}\mathcal{G})]/\delta g^{\mu\nu}$ for $\delta S_{\text{LBD}}/\delta g^{\mu\nu}$ are no longer pure divergences, because the scalar field $\phi(x^\alpha)$ as a nontrivial coefficient will be absorbed into the variations of *RR and $\sqrt{-g}\mathcal{G}$ when integrating by parts. Hence, although S_L and S_{HE} are indistinguishable, their respective scalar-

¹ Note that not to confuse with the more common "Lovelock action" for the topological generalizations of the Hilbert-Einstein action to generic N dimensions that still preserves second-order field equations, as in Ref.[6].

tensor counterparts \mathcal{S}_{LBD} and \mathcal{S}_{BD} are different.

Note that the cosmological-constant term -2Λ in Eq.(2) is temporarily abandoned in \mathcal{L}_L ; otherwise, it would add an extra term $-2\Lambda\phi$ to \mathcal{L}_{LBD} , which serves as a simplest linear potential. This is primarily for a better analogy between the LBD and the Brans-Dicke gravities, as the latter in its standard form does not contain a potential term $V(\phi)$, and an unspecified potential $V(\phi)$ would cause too much arbitrariness to \mathcal{L}_{LBD} .

Also, Lovelock's original action Eq.(2) concentrates on the algebraic curvature invariants; in fact, one can further add to Eq.(2) the relevant differential terms $\square R$, $\square^* RR$, and $\square \mathcal{G}$ ($\square = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ denoting the covariant d'Alembertian), while the field equation will remain unchanged. This way, the gravitational Lagrangian density in Eq.(7) is enriched into

$$\mathcal{L} = \frac{1}{16\pi G} \left(R + \frac{a^* RR}{\sqrt{-g}} + b\mathcal{G} + c\square R + d\square^* RR + e\square \mathcal{G} \right), \quad (14)$$

with $\{c, d, e\}$ being constants, and its Brans-Dicke-type counterpart extends Eq.(12) into

$$\mathcal{L} = \frac{1}{16\pi} \left[\phi \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) - \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi + \phi \left(c\square R + d\square^* RR + e\square \mathcal{G} \right) \right], \quad (15)$$

where $\phi \cdot (c\square R + d\square^* RR + e\square \mathcal{G})$ have nontrivial contributions to the field equation. However, unlike *RR and $\sqrt{-g}\mathcal{G}$ which are divergences of their respective topological current as in Eqs.(4) and (5), $\{\square R, \square^* RR, \square \mathcal{G}\}$ are total derivatives simply because the d'Alembertian \square satisfies $\sqrt{-g}\square\Theta = \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta \Theta)$ when acting on an arbitrary scalar field Θ ; in this sense, these differential boundary terms which contain fourth-order derivatives of the metric are less interesting than *RR and \mathcal{G} . In this paper, we will focus on the LBD gravity \mathcal{L}_{LBD} Eq.(12) built upon the original Lovelock action and Lovelock's theorem, rather than Eq.(15) out of the modified action Eq.(14).

III. GRAVITATIONAL AND WAVE EQUATIONS

In this section we will work out the gravitational field equation $\delta \mathcal{S}_{\text{LBD}} / \delta g^{\mu\nu} = 0$ and the wave equation $\delta \mathcal{S}_{\text{LBD}} / \delta \phi = 0$ for the LBD gravity. First of all, with $\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}$, $\delta \Gamma_{\alpha\beta}^\lambda = \frac{1}{2} g^{\lambda\sigma} (\nabla_\alpha \delta g_{\sigma\beta} + \nabla_\beta \delta g_{\sigma\alpha} - \nabla_\sigma \delta g_{\alpha\beta})$, and the Palatini identity $\delta R_{\alpha\beta\gamma}^\lambda = \nabla_\beta (\delta \Gamma_{\gamma\alpha}^\lambda) - \nabla_\gamma (\delta \Gamma_{\beta\alpha}^\lambda)$ [23], for the first term ϕR in \mathcal{L}_{LBD} it is easy to work out that

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \phi R)}{\delta g^{\mu\nu}} \cong -\frac{1}{2} \phi R g_{\mu\nu} + \phi R_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \phi, \quad (16)$$

where \cong means equality by neglecting all total-derivative terms which are boundary terms for the action.

III.1. Coupling to the Chern-Pontryagin invariant

The Chern-Pontryagin density *RR in \mathcal{L}_{LBD} measures the gravitational effects of parity violation through $\int d^4x \phi {}^*RR$ for its dependence on the Levi-Civita pseudotensor. In addition to Eq.(8), *RR is related to the left dual of the Riemann tensor via

$${}^*RR = \frac{1}{2} (\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}{}_{\gamma\delta}) R^{\alpha\beta\gamma\delta} = {}^*R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}. \quad (17)$$

Applying the Ricci decomposition $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2} (g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\beta\gamma} + g_{\beta\delta} R_{\alpha\gamma} - g_{\beta\gamma} R_{\alpha\delta}) - \frac{1}{6} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) R$ to Eq.(17) and using the cyclic identity $C_{\alpha\beta\gamma\delta} + C_{\alpha\gamma\delta\beta} + C_{\alpha\delta\beta\gamma} = 0$ for the traceless Weyl tensor, one could find the equivalence

$${}^*RR = {}^*CC := \frac{1}{2} (\epsilon_{\alpha\beta\mu\nu} C^{\mu\nu}{}_{\gamma\delta}) C^{\alpha\beta\gamma\delta} = {}^*C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \quad (18)$$

which indicates that the Chern-Pontryagin density is conformally invariant [15] under a rescaling $g_{\mu\nu} \mapsto \Omega(x^\alpha)^2 \cdot g_{\mu\nu}$ of the metric tensor.

With the Chern-Simons topological current K^μ in Eq.(4), one can integrate by parts and obtain $\int d^4x \phi {}^*RR = -4 \int d^4x \phi (\partial_\mu K^\mu) = -4 \int d^4x \partial_\mu (\phi K^\mu) + 4 \int d^4x (\partial_\mu \phi) K^\mu$. Hence, instead of directly varying $\phi {}^*RR$ with respect to the inverse metric, we firstly vary the four-current K^μ by the Levi-Civita connection. It follows that

$$\begin{aligned} \delta \int d^4x \phi {}^*RR &\cong 4 \int d^4x (\partial_\mu \phi) \delta K^\mu \\ &= 2 \int d^4x (\partial_\mu \phi) \epsilon^{\mu\alpha\beta\gamma} R^{\xi}{}_{\rho\beta\gamma} \delta \Gamma_{\alpha\xi}^\rho \\ &= 2 \int d^4x (\partial_\mu \phi) \epsilon^{\mu\alpha\beta\gamma} R^{\xi\nu}{}_{\beta\gamma} (\nabla_\xi \delta g_{\alpha\nu} - \nabla_\nu \delta g_{\alpha\xi}) \\ &\cong -2 \int d^4x [(\partial_\mu \phi) \epsilon^{\mu\alpha\beta\gamma} \nabla_\xi R^{\xi\nu}{}_{\beta\gamma} + (\partial_\mu \partial_\xi \phi) \epsilon^{\mu\alpha\beta\gamma} R^{\xi\nu}{}_{\beta\gamma}] \delta g_{\alpha\nu} \\ &= -4 \int d^4x [(\partial_\mu \phi) \epsilon^{\mu\alpha\beta\gamma} \nabla_\beta R_{\gamma}{}^\nu{}_\nu + (\partial_\mu \partial_\xi \phi) {}^*R^{\mu\alpha\xi\nu}] \delta g_{\alpha\nu} \quad (19) \\ &= 4 \int d^4x [(\partial^\mu \phi) \epsilon_{\mu\alpha\beta\gamma} \nabla^\beta R_{\gamma}{}^\nu{}_\nu + (\partial_\mu \partial_\xi \phi) {}^*R^{\mu}{}_{\alpha}{}^\xi{}_\nu] \delta g^{\alpha\nu}, \quad (20) \end{aligned}$$

where, in the third row we expanded $\delta \Gamma_{\alpha\xi}^\rho$ and made use of the cancelation $R^{\xi\nu}{}_{\beta\gamma} \nabla_\alpha \delta g_{\xi\nu} = 0$ due to the skew-symmetry for the indices $\xi \leftrightarrow \nu$; in the fourth row, we applied the replacement $\nabla_\xi R^{\xi\nu}{}_{\beta\gamma} = \nabla_\beta R_{\gamma}{}^\nu{}_\nu - \nabla_\gamma R_{\beta}{}^\nu{}_\nu$ in accordance with the relation

$$\nabla^\alpha R_{\alpha\mu\beta\nu} = \nabla_\beta R_{\mu\nu} - \nabla_\nu R_{\mu\beta}, \quad (21)$$

which is an implication of the second Bianchi identity $\nabla_\gamma R_{\alpha\mu\beta\nu} + \nabla_\nu R_{\alpha\mu\gamma\beta} + \nabla_\beta R_{\alpha\mu\nu\gamma} = 0$; in the last step, we raised the indices of $\delta g_{\alpha\nu}$ to $\delta g^{\alpha\nu}$ and thus had the overall minus sign dropped. In Eq.(20) we adopted the usual notation $\partial^\mu \phi \equiv g^{\hat{\mu}\mu} \partial_{\hat{\mu}} \phi$, and note that $(\partial_\mu \partial_\xi \phi) {}^*R^{\mu}{}_{\alpha}{}^\xi{}_\nu \neq (\partial^\mu \partial^\xi \phi) {}^*R_{\mu\xi\nu}{}^\alpha$ since in general the metric tensor does not commute with partial derivatives and thus $\partial^\mu \partial^\xi \phi = g^{\hat{\mu}\hat{\mu}} \partial_{\hat{\mu}} (g^{\hat{\xi}\hat{\xi}} \partial_{\hat{\xi}} \phi) \neq g^{\hat{\mu}\hat{\mu}} g^{\hat{\xi}\hat{\xi}} \partial_{\hat{\mu}} \partial_{\hat{\xi}} \phi$. Relabel the in-

dices of Eq.(19) and we obtain the variational derivative

$$\frac{1}{\sqrt{-g}} \frac{\delta(\phi^* RR)}{\delta g^{\mu\nu}} =: H_{\mu\nu}^{(\text{CP})} \quad \text{and}$$

$$\sqrt{-g} H_{\mu\nu}^{(\text{CP})} = 2\partial^\xi \phi \cdot (\epsilon_{\xi\mu\alpha\beta} \nabla^\alpha R^\beta{}_\nu + \epsilon_{\xi\nu\alpha\beta} \nabla^\alpha R^\beta{}_\mu) + 2\partial_\alpha \partial_\beta \phi \cdot (*R^\alpha{}_\mu{}^\beta{}_\nu + *R^\alpha{}_\nu{}^\beta{}_\mu). \quad (22)$$

Compared with Eq.(16), $H_{\mu\nu}^{(\text{CP})}$ does not contain a $-\frac{1}{2}\phi^* RR g_{\mu\nu}$ term, because $*RR$ by itself already serves as a covariant density as opposed to the usual form $\sqrt{-g}\mathcal{R}$ for other curvature invariants.

Note that the nonminimal coupling between a scalar field and $*RR$ is crucial to the Chern-Simons gravity; however, its original proposal Ref. [11] had adopted the opposite geometric system which uses the metric signature $(+, - - -)$, the conventions $\{R^\alpha{}_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha{}_{\gamma\beta} \cdots, R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}\}$, Einstein's equation $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}^{(\text{m})}$, and the definition $*RR = -*R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = -\frac{1}{2}(\epsilon_{\alpha\beta\mu\nu} R^{\mu\nu}{}_{\gamma\delta}) R^{\alpha\beta\gamma\delta}$. This has caused quite a few mistakes in the subsequent Chern-Simons literature that adopt different conventions, and we hope the details in this subsection could correct these misunderstandings. Also, in Eq. (22), the quantities $\{\epsilon_{\xi\mu\alpha\beta}, K^\mu, *RR, R^\beta{}_\nu, *R_{\beta\mu\alpha\nu}\}$ have the same values in both sets of sign conventions. See our note Ref.[24] for further clarification of this issue.

III.2. Coupling to the Gauss-Bonnet invariant

The third term $\phi\mathcal{G}$ in \mathcal{L}_{LBD} represents the nonminimal coupling between the scalar field and the Gauss-Bonnet invariant $\mathcal{G} = R^2 - 4R_c^2 + R_m^2$, where we have employed the straightforward abbreviations $R_c^2 := R_{\alpha\beta} R^{\alpha\beta}$ and $R_m^2 := R_{\alpha\mu\beta\nu} R^{\alpha\mu\beta\nu}$ to denote the Ricci and Riemann tensor squares. Following the standard procedures of variational derivative as before in $\delta(\sqrt{-g}\phi R)/\delta g^{\mu\nu}$, we have

$$\frac{\delta(\sqrt{-g}\phi\mathcal{G})}{\sqrt{-g}\delta g^{\mu\nu}} = \frac{\delta(\phi R^2)}{\delta g^{\mu\nu}} - 4\frac{\delta(\phi R_c^2)}{\delta g^{\mu\nu}} + \frac{\delta(\phi R_m^2)}{\delta g^{\mu\nu}} - \frac{1}{2}\phi\mathcal{G}g_{\mu\nu}, \quad (23)$$

with

$$\frac{\delta(\phi R^2)}{\delta g^{\mu\nu}} \cong 2\phi RR_{\mu\nu} + 2(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)(\phi R) \quad (24)$$

$$\frac{\delta(\phi R_c^2)}{\delta g^{\mu\nu}} \cong 2\phi R_\mu{}^\alpha R_{\alpha\nu} + \square(\phi R_{\mu\nu}) - \nabla_\alpha \nabla_\nu(\phi R_\mu{}^\alpha) - \nabla_\alpha \nabla_\mu(\phi R_\nu{}^\alpha) + g_{\mu\nu} \nabla_\alpha \nabla_\beta(\phi R^{\alpha\beta}) \quad (25)$$

$$\frac{\delta(\phi R_m^2)}{\delta g^{\mu\nu}} \cong 2\phi R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} + 4\nabla^\beta \nabla^\alpha(\phi R_{\alpha\mu\beta\nu}), \quad (26)$$

where total-derivative terms have been removed. Recall that besides Eq.(21), the second Bianchi identity also has the following implications which transform the derivative of a high-

rank curvature tensor into that of lower-rank tensors plus non-linear algebraic terms:

$$\nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \nabla_\beta R \quad (27)$$

$$\nabla^\beta \nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \square R \quad (28)$$

$$\nabla^\beta \nabla^\alpha R_{\alpha\mu\beta\nu} = \square R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R + R_{\alpha\mu\beta\nu} R^{\alpha\beta} - R_\mu{}^\alpha R_{\alpha\nu} \quad (29)$$

$$\nabla^\alpha \nabla_\mu R_{\alpha\nu} + \nabla^\alpha \nabla_\nu R_{\alpha\mu} = \nabla_\mu \nabla_\nu R - 2R_{\alpha\mu\beta\nu} R^{\alpha\beta} + 2R_\mu{}^\alpha R_{\alpha\nu} \quad (30)$$

Using Eq.(21) and Eqs.(27)-(30) to expand the second-order covariant derivatives in Eqs.(24)-(26), and putting them back into Eq.(23), we obtain

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\phi\mathcal{G})}{\delta g^{\mu\nu}} =: H_{\mu\nu}^{(\text{GB})} \quad \text{with}$$

$$H_{\mu\nu}^{(\text{GB})} = \phi(2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu} R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma}) + 2R(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)\phi + 4R_\mu{}^\alpha \nabla_\alpha \nabla_\nu \phi + 4R_\nu{}^\alpha \nabla_\alpha \nabla_\mu \phi - 4R_{\mu\nu}\square\phi - 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + 4R_{\alpha\mu\beta\nu} \nabla^\beta \nabla^\alpha \phi - \frac{1}{2}\phi\mathcal{G}g_{\mu\nu}, \quad (31)$$

where the second-order derivatives $\{\square, \nabla_\alpha \nabla_\nu, \text{etc}\}$ only act on the scalar field ϕ .

However, we realize that Eq.(31) is still not the ultimate expression. In four dimensions, $\sqrt{-g}\mathcal{G}$ is proportional to the Euler-Poincaré topological density, $\mathcal{G} = (\frac{1}{2}\epsilon_{\alpha\beta\gamma\zeta} R^{\gamma\zeta\eta\xi}) \cdot (\frac{1}{2}\epsilon_{\eta\xi\rho\sigma} R^{\rho\sigma\alpha\beta}) = *R_{\alpha\beta}{}^{\eta\xi} \cdot *R_{\eta\xi}{}^{\alpha\beta}$, and the integral $\frac{1}{32\pi^2} \int d^4x \sqrt{-g}\mathcal{G}$ equates the Euler characteristic $\chi(\mathcal{M})$ of the spacetime. Thus $\frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g}\mathcal{G} = 32\pi^2 \frac{\delta}{\delta g^{\mu\nu}} \chi(\mathcal{M}) \equiv 0$. Based on Eqs.(24)-(26), one could easily obtain the Bach-Lanczos identity from the explicit variational derivative $\delta(\sqrt{-g}\mathcal{G})/\delta g^{\mu\nu}$,

$$2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu} R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} \equiv \frac{1}{2}\mathcal{G}g_{\mu\nu}, \quad (32)$$

with which Eq.(31) can be best simplified into

$$H_{\mu\nu}^{(\text{GB})} = 2R(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)\phi + 4R_\mu{}^\alpha \nabla_\alpha \nabla_\nu \phi + 4R_\nu{}^\alpha \nabla_\alpha \nabla_\mu \phi - 4R_{\mu\nu}\square\phi - 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + 4R_{\alpha\mu\beta\nu} \nabla^\beta \nabla^\alpha \phi, \quad (33)$$

whose trace is

$$g^{\mu\nu} H_{\mu\nu}^{(\text{GB})} = 2R\square\phi - 4R^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi. \quad (34)$$

In the existent literature, the effects of the generalized and thus nontrivial Gauss-Bonnet dependence for the field equations are generally depicted in the form analogous to Eq.(31), such as the string-inspired Gauss-Bonnet effective dark energy [12] with $\mathcal{L} = \frac{1}{16\pi G} R - \frac{\gamma}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) + f(\varphi)\mathcal{G}$, as well as the $R + f(\mathcal{G})$ [3], the $f(R, \mathcal{G})$ [4] and the $f(R, \mathcal{G}, \mathcal{L}_m)$ [14] generalized Gauss-Bonnet gravities. Here we emphasize that the Gauss-Bonnet effects therein could all be simplified into

the form of Eq.(33).

III.3. Gravitational field equation

Collecting the results in Eqs.(16), (22), and (33), we finally obtain the gravitational field equation

$$\begin{aligned} \phi \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) - \frac{\omega_L}{\phi} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) \\ + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \phi + a H_{\mu\nu}^{(\text{CP})} + b H_{\mu\nu}^{(\text{GB})} = 8\pi T_{\mu\nu}^{(\text{m})}, \end{aligned} \quad (35)$$

where $H_{\mu\nu}^{(\text{CP})}$ vanishes for all spherically symmetric or conformal flat spacetimes. Eq.(35) yields the trace equation

$$-\phi R + \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi + (3 + 2bR) \square \phi - 4bR^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 8\pi T^{(\text{m})}, \quad (36)$$

where $H_{\mu\nu}^{(\text{CP})}$ is always traceless, $g^{\mu\nu} H_{\mu\nu}^{(\text{CP})} \equiv 0$ – this is not a surprise because it equivalently traces back to the effects of the dual square *CC of the traceless Weyl tensor.

Note that in existent studies the invariants *RR and \mathcal{G} have demonstrated their importance in various aspects. For example, as shown by Eq.(6) of Ref.[25] [recall the equivalence ${}^*RR = {}^*CC$ in Eq.(18)], in the effective field theory for the initial cosmic inflation, the only leading-order fluctuations to the standard inflation action in the tensor modes are the parity-violation Chern-Pontryagin and the topological Gauss-Bonnet effects.

III.4. Wave equations

Straightforward extremization of \mathcal{S}_{LBD} with respect to the scalar field yields the *kinematical* wave equation

$$\frac{2\omega_L}{\phi} \square \phi = -R + \frac{\omega_L}{\phi^2} \nabla_\alpha \phi \nabla^\alpha \phi - \left(\frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right), \quad (37)$$

with $\square \phi = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta \phi)$. We regard Eq.(37) as “kinematical” because it does not explicitly relate the propagation of ϕ to the matter distribution \mathcal{L}_m or $T^{(\text{m})} = g^{\mu\nu} T_{\mu\nu}^{(\text{m})}$.

Combine Eq.(37) with the gravitational trace equation (36), and it follows that

$$\begin{aligned} (2\omega_L + 3 + 2bR) \square \phi = - \left(\frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) \phi \\ + 8\pi T^{(\text{m})} + 4bR^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi, \end{aligned} \quad (38)$$

which serves as the generalized Klein-Gordon equation that governs the dynamics of the scalar field.

IV. THE $\omega_L \rightarrow \infty$ LIMIT AND GR

From the dynamical equation (38), we obtain

$$\begin{aligned} \square \phi = \frac{1}{2\omega_L + 3 + 2bR} \left\{ - \left(\frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) \phi \right. \\ \left. + 8\pi T^{(\text{m})} + 4bR^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi \right\}. \end{aligned} \quad (39)$$

The topology-gravity coupling strengths $\{a, b\}$ should take finite values – just like the Newtonian constant G for matter-gravity coupling. Similarly the curvature invariants $\{R, {}^*RR, \mathcal{G}\}$ for a physical spacetime should be finite, and we further assume the scalar field ϕ to be nonsingular. Thus, in the limit $\omega_L \rightarrow \infty$, Eq.(39) yields $\square \phi = \mathcal{O}\left(\frac{1}{\omega_L}\right)$ and

$$\phi = \langle \phi \rangle + \mathcal{O}\left(\frac{1}{\omega_L}\right) = \frac{1}{G} + \mathcal{O}\left(\frac{1}{\omega_L}\right), \quad (40)$$

where $\langle \phi \rangle$ denotes the expectation value of the scalar field and we expect it to be the inverse of the Newtonian constant $1/G$. Under the behaviors Eq.(40) in the infinite ω_L limit, we have $H_{\mu\nu}^{(\text{CP})} = 0 = H_{\mu\nu}^{(\text{GB})}$, and the field equation (35) reduces to become Einstein’s equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}^{(\text{m})}$.

On the other hand, from Eq.(39) we can also observe that $\square \phi \equiv 0$ in the special situation

$$-4bR^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + \left(\frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right) \phi = 8\pi T^{(\text{m})}, \quad (41)$$

and the scalar field becomes undeterminable from the dynamical equation (39).

The term $-4bR^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi$ comes from the trace $g^{\mu\nu} H_{\mu\nu}^{(\text{GB})}$, while *RR and \mathcal{G} are respectively related to the topological instanton number [15] and the Euler characteristic. Thus, all terms on the left hand side of Eq.(41) are related to topological effects nonminimally coupled with ϕ , and they cancel out the trace of the matter tensor. In this sense, we call Eq.(41) the *topological balance condition*.

Putting $\square \phi \equiv 0$ and the condition Eq.(41) back into the trace equation (36), we obtain

$$\begin{aligned} \frac{\omega_L}{\phi^2} \nabla_\alpha \phi \nabla^\alpha \phi = R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \\ = -\omega_L \square \ln \phi, \end{aligned} \quad (42)$$

where in the second step we further made use of the expansion $\square \ln \phi = \nabla^\alpha \left(\frac{1}{\phi} \nabla_\alpha \phi \right) = -\frac{1}{\phi^2} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{\phi} \square \phi = -\frac{1}{\phi^2} \nabla_\alpha \phi \nabla^\alpha \phi$ for $\square \phi \equiv 0$. Thus it follows that

$$\omega_L \nabla_\alpha (\ln \phi) \nabla^\alpha (\ln \phi) = R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G}. \quad (44)$$

For $\omega_L \rightarrow \infty$, this equation gives the estimate

$$\|\nabla_\alpha (\ln \phi)\| \sim \sqrt{\frac{R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G}}{\omega_L}} \sim \mathcal{O}\left(\frac{1}{\sqrt{\omega_L}}\right), \quad (45)$$

which integrates to yield $\ln \phi = \text{constant} + \mathcal{O}\left(\frac{1}{\sqrt{\omega_L}}\right)$. Hence, ϕ satisfies

$$\phi \sim \phi_0 + \mathcal{O}\left(\frac{1}{\sqrt{\omega_L}}\right), \quad (46)$$

where the constant ϕ_0 is the average value of ϕ . In accordance with Eq.(42) and the estimate Eq.(46), the term $-\frac{\omega_L}{\phi}(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi)$ in the field equation (35), which arises from the source $-\frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi$ in S_{LBD} , will not vanish. This way, the $\omega_L \rightarrow \infty$ limit could not recover Einstein's equation and GR in situations where the topological balance condition Eq.(41) holds, although the existence of such solutions remains to be carefully checked.

This is similar to the Brans-Dicke theory given by the action Eq.(10), which recovers GR in the limit $\omega_{\text{BD}} \rightarrow \infty$, unless the stress-energy-momentum tensor has a vanishing trace $T^{(m)} = 0$ [26], such as the matter content being radiation with $P_{\text{rad}} = \frac{1}{3}\rho_{\text{rad}}$ and $T_{\text{rad}}^{(m)} = -\rho_{\text{rad}} + 3P_{\text{rad}} = 0$.

V. ENERGY-MOMENTUM CONSERVATION

In modified gravities with the generic Lagrangian density $\mathcal{L} = f(R, \mathcal{R}_i, \dots)$, where $\mathcal{R}_i = \mathcal{R}_i(g_{\alpha\beta}, R_{\alpha\mu\beta\nu}, \nabla_\gamma R_{\alpha\mu\beta\nu}, \dots, \nabla_{\gamma_1} \nabla_{\gamma_2} \dots \nabla_{\gamma_n} R_{\alpha\mu\beta\nu})$ and the “...” in $\mathcal{L} = f$ refer to arbitrary curvature invariants beyond the Ricci scalar, the energy-momentum conservation is naturally guaranteed by Noether's law or the generalized contracted Bianchi identities [27]

$$\nabla^\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta [\sqrt{-g} f(R, \mathcal{R}_i, \dots)]}{\delta g^{\mu\nu}} \right) = 0, \quad (47)$$

which can be expanded into

$$f_R R_{\mu\nu} + \sum f_{\mathcal{R}_i} \mathcal{R}_{\mu\nu}^{(i)} - \frac{1}{2} f(R, \mathcal{R}_i, \dots) g_{\mu\nu} = 0, \quad (48)$$

where $f_R := \partial f(R, \mathcal{R}_i, \dots) / \partial R$, $f_{\mathcal{R}_i} := \partial f(R, \mathcal{R}_i, \dots) / \partial \mathcal{R}_i$, and $\mathcal{R}_{\mu\nu}^{(i)} \cong (f_{\mathcal{R}_i} \delta \mathcal{R}_i) / \delta g^{\mu\nu}$. However, in the more generic situations of scalar-tensor-type gravities with $\mathcal{L} = f(\phi, R, \mathcal{R}_i, \dots) + \varpi(\phi, \nabla_\alpha \phi \nabla^\alpha \phi)$ where nonminimal couplings between the scalar fields and the curvature invariants are involved, such as the LBD proposal under discussion, the conservation problem is more complicated than pure tensorial gravity.

Now let's get back to the LBD field equation (35). By the coordinate invariance or the diffeomorphism invariance of the matter action \mathcal{S}_m in which \mathcal{L}_m is neither coupled with the curvature invariants nor the scalar field ϕ , naturally we have the energy-momentum conservation $\nabla^\mu T_{\mu\nu}^{(m)} = 0$ for the matter content. Thus, the covariant derivative of the left hand side of Eq.(35) should also vanish. With the Bianchi identity $\nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$ and the third-order-derivative commutator $(\nabla_\nu \square - \square \nabla_\nu) \phi = -R_{\mu\nu} \nabla^\mu \phi$, it follows that

$$\nabla^\mu \left[\phi \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \phi \right] = -\frac{1}{2} R \nabla_\nu \phi. \quad (49)$$

Moreover, for the scalar field, we have

$$\begin{aligned} & \nabla^\mu \left[-\frac{\omega_L}{\phi} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) \right] \\ &= \frac{1}{2} \nabla_\nu \phi \cdot \left(\frac{\omega_L}{\phi^2} \nabla_\alpha \phi \nabla^\alpha \phi - \frac{2\omega_L}{\phi} \square \phi \right) \\ &= \frac{1}{2} \nabla_\nu \phi \cdot \left(R + \frac{a}{\sqrt{-g}} {}^*RR + b\mathcal{G} \right), \end{aligned} \quad (50)$$

where the kinematical wave equation (37) has been employed.

For the Chern-Pontryagin and the Gauss-Bonnet parts in Eq.(35), consider the componential actions $\mathcal{S}_{\text{CP}} = \int d^4x \phi {}^*RR$ and $\mathcal{S}_{\text{GB}} = \int d^4x \sqrt{-g} \phi \mathcal{G}$. Under an arbitrary infinitesimal coordinate transformation $x^\mu \mapsto x^\mu + \delta x^\mu$, where $\delta x^\mu = \xi^\mu$ is an infinitesimal vector field which vanishes on the boundary, so that the spacetime manifold is mapped onto itself. Then \mathcal{S}_{CP} and \mathcal{S}_{GB} vary by

$$\delta \mathcal{S}_{\text{CP}} = - \int d^4x \phi \partial_\mu (\xi^\mu {}^*RR) \cong \int d^4x {}^*RR (\partial_\mu \phi) \xi^\mu, \quad (51)$$

$$\delta \mathcal{S}_{\text{GB}} = - \int d^4x \phi \partial_\mu (\xi^\mu \sqrt{-g} \mathcal{G}) \cong \int d^4x \sqrt{-g} \mathcal{G} (\partial_\mu \phi) \xi^\mu. \quad (52)$$

For the first step in Eqs.(51) and Eqs.(52), one should note that $x^\mu \mapsto x^\mu + \xi^\mu$ is a particle/active transformation, under which the dynamical tensor fields transform, while the background scalar field $\phi(x^\alpha)$ and the coordinate system parameterizing the spacetime manifold remain unchanged [28]. On the other hand, the inverse metric transforms by $g^{\mu\nu} \mapsto g^{\mu\nu} + \delta g^{\mu\nu}$ with $\delta g^{\mu\nu} = -\mathcal{L}_\xi g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$, and thus we have

$$\begin{aligned} \delta \mathcal{S}_{\text{CP}} &= 2 \int d^4x \sqrt{-g} H_{\mu\nu}^{(\text{CP})} \nabla^\mu \xi^\nu \\ &\cong -2 \int d^4x \sqrt{-g} (\nabla^\mu H_{\mu\nu}^{(\text{CP})}) \xi^\nu, \end{aligned} \quad (53)$$

$$\begin{aligned} \delta \mathcal{S}_{\text{GB}} &= 2 \int d^4x \sqrt{-g} H_{\mu\nu}^{(\text{GB})} \nabla^\mu \xi^\nu \\ &\cong -2 \int d^4x \sqrt{-g} (\nabla^\mu H_{\mu\nu}^{(\text{GB})}) \xi^\nu. \end{aligned} \quad (54)$$

Comparing Eqs.(51) with (53), and Eqs.(52) with (54), we obtain the relations

$$\nabla^\mu H_{\mu\nu}^{(\text{CP})} = -\frac{1}{2} \frac{{}^*RR}{\sqrt{-g}} \cdot \partial_\nu \phi, \quad (55)$$

$$\nabla^\mu H_{\mu\nu}^{(\text{GB})} = -\frac{1}{2} \mathcal{G} \cdot \partial_\nu \phi. \quad (56)$$

Adding up Eqs.(49), (50), (55), and (56), one could find that the covariance divergence for the left hand side of the field equation (35) vanishes, which confirms the energy-momentum conservation in the LBD gravity.

Eqs.(55) and (56) for the nontrivial divergences of $H_{\mu\nu}^{(\text{CP})}$ and $H_{\mu\nu}^{(\text{GB})}$, by their derivation process, reflect the breakdown of diffeomorphism invariance for \mathcal{S}_{CP} and \mathcal{S}_{GB} in S_{LBD} . They

have clearly shown the influences of nonminimal ϕ -topology couplings to the covariant conservation, as opposed to the straightforward generalized Bianchi identities

$$\nabla^\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta^* RR}{\delta g^{\mu\nu}} \right) = 0 \quad \text{and} \quad \nabla^\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{G})}{\delta g^{\mu\nu}} \right) = 0. \quad (57)$$

VI. CONFORMAL TRANSFORMATIONS

The standard LBD action \mathcal{S}_{LBD} in Eq.(12) can be transformed into different representations by conformal rescaling of the spacetime line element, which geometrically preserves the angles between spacetime vectors and physically retains local causality structures.

VI.1. Dynamical Chern-Simons gravity

As a simplest example, consider the specialized \mathcal{S}_{LBD} in vacuum and for spacetimes of negligible gravitational effects from the nonminimally ϕ -coupled Gauss-Bonnet term. With $\mathcal{S}_m = 0$ and $b = 0$, Eq.(12) reduces to become

$$S = \frac{1}{16\pi} \int d^4x \left[\sqrt{-g} \left(\phi R - \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right) + a \phi^* RR \right]. \quad (58)$$

For a pointwise scaling field $\Omega = \Omega(x^\alpha) > 0$, we can rescale the metric $g_{\mu\nu}$ of the original frame into $\tilde{g}_{\mu\nu}$ via $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$; it follows that $g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}$, $g^{\mu\nu} = \Omega^2 \tilde{g}^{\mu\nu}$, $\sqrt{-g} = \Omega^{-4} \sqrt{-\tilde{g}}$, and² [10]

$$R = \Omega^2 \left[\tilde{R} + 6\tilde{\square}(\ln \Omega) - 6\tilde{g}^{\alpha\beta} \partial_\alpha(\ln \Omega) \partial_\beta(\ln \Omega) \right]. \quad (59)$$

Hence, for the reduced LBD action Eq.(58), the conformal transformation

$$g_{\mu\nu} = \frac{1}{G\phi} \cdot \tilde{g}_{\mu\nu} \quad (60)$$

along with the redefinition of the scalar field $\{\vartheta = \vartheta(x^\alpha), \phi = \phi(\vartheta)\}$ lead to

$$S \cong \frac{1}{16\pi G} \int d^4x \left[\sqrt{-\tilde{g}} \left(\tilde{R} - \frac{2\omega_L + 3}{2\phi(\vartheta)^2} \left(\frac{d\phi}{d\vartheta} \right)^2 \tilde{\nabla}_\alpha \vartheta \tilde{\nabla}^\alpha \vartheta \right) + a \phi(\vartheta)^* RR \right], \quad (61)$$

where the scalar field ϑ no longer directly couples to the Ricci scalar \tilde{R} , and thus the $6\tilde{\square}(\ln \Omega)$ component in Eq.(59) has been removed as it simply yields a boundary term $6 \int \partial_\alpha \left[\sqrt{-\tilde{g}} \partial^\alpha (\ln \Omega) \right] d^4x$ for the action. Also, Eq.(61) has uti-

lized the fact that the $(1, 3)$ -type Weyl tensor $C^\alpha_{\beta\gamma\delta}$ and thus ${}^*RR = {}^*CC = {}^*\widetilde{CC} = {}^*\widetilde{RR}$ are conformally invariant. It is straightforward to observe from Eq.(61) that the kinetics of ϑ is canonical for $\omega_L > -3/2$, noncanonical for $\omega_L < -3/2$, and nondynamical for $\omega_L = -3/2$; here we are interested in the canonical case. For the specialization

$$d\vartheta = \pm \sqrt{2\omega_L + 3} \frac{d\phi}{\phi}, \quad (62)$$

which integrates to yield

$$\vartheta = \pm \sqrt{2\omega_L + 3} \ln \frac{\phi}{\phi_0}, \quad (63)$$

where ϕ_0 is an integration constant, or inversely

$$\phi = \phi_0 \exp \left(\pm \frac{\vartheta}{\sqrt{2\omega_L + 3}} \right), \quad (64)$$

the action Eq.(65) finally becomes

$$S = \frac{1}{16\pi G} \int d^4x \left[\sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2} \tilde{\nabla}_\alpha \vartheta \tilde{\nabla}^\alpha \vartheta \right) + a \phi_0 \exp \left(\pm \frac{\vartheta}{\sqrt{2\omega_L + 3}} \right) {}^*\widetilde{RR} \right]. \quad (65)$$

Hence, the conformal rescaling $g_{\mu\nu} = \tilde{g}_{\mu\nu}/G\phi$ along with the new scalar field $\vartheta(x^\alpha)$ recast the reduced LBD action Eq.(58) into Eq.(65), which is an action for the dynamical Chern-Simons gravity [19], though the nonminimal ϑ - *RR coupling is slightly more complicated than the straightforward ϑ - *RR as in the popular Chern-Simons literature. Moreover, the conformal invariance of *RR guarantees that the effect of $\int d^4x \phi^* RR$ could never be removed by conformal transformations.

Note that the matter action $\mathcal{S}_m(g_{\mu\nu}, \psi_m)$ would be transformed into $\mathcal{S}_m(\tilde{g}_{\mu\nu}/G\phi, \psi_m)$ (in general \mathcal{S}_m does not contain derivatives of the metric tensor [20]), which are different in the ϕ - \mathcal{S}_m or ϕ - \mathcal{L}_m couplings; consequently $T_{\mu\nu}^{(m)}$ fails to be conformally invariant unless it is traceless $T^{(m)} = 0$ [10]. This is why we focus on the vacuum situation.

VI.2. Generalized Gauss-Bonnet dark energy

Similarly, in vacuum and for spacetimes of negligible Chern-Simons parity-violation effect, \mathcal{S}_{LBD} reduces into

$$S = \frac{1}{16\pi} \int d^4x \left[\sqrt{-g} \left(\phi R + b\phi \mathcal{G} - \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right) \right]. \quad (66)$$

Under the local rescaling $g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ for the metric, the Gauss-Bonnet scalar satisfies [29]

² Compared with $R = \Omega^2 [\tilde{R} + 6\tilde{\square}(\ln \Omega) - 12\tilde{g}^{\alpha\beta} \partial_\alpha \ln \Omega \partial_\beta \ln \Omega]$, Eq.(59) best isolates pure-divergence terms and thus most simplifies the action once the

coefficient of R is reset into unity. Moreover, by employing $\ln \Omega$ instead of Ω , the transformations $R \rightarrow \tilde{R}$ becomes skew-symmetric to $\tilde{R} \rightarrow R$.

$$\mathcal{G} = \Omega^4 \left\{ \tilde{\mathcal{G}} - 8\tilde{R}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\ln\Omega) - 8\tilde{R}^{\alpha\beta}\tilde{\nabla}_\alpha(\ln\Omega)\tilde{\nabla}_\beta(\ln\Omega) + 4\tilde{R}\tilde{\Box}(\ln\Omega) + 8[\tilde{\Box}(\ln\Omega)]^2 - 8\tilde{\Box}(\ln\Omega)\cdot\tilde{\nabla}_\alpha(\ln\Omega)\tilde{\nabla}^\alpha(\ln\Omega) \right. \\ \left. - 8\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\ln\Omega)\cdot\tilde{\nabla}^\alpha\tilde{\nabla}^\beta(\ln\Omega) - 16\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\ln\Omega)\cdot\tilde{\nabla}^\alpha(\ln\Omega)\tilde{\nabla}^\beta(\ln\Omega) \right\}. \quad (67)$$

Set the factor of conformal transformation to be $\Omega = \sqrt{G}\phi$ so that the Ricci scalar decouples from the scalar field, and redefine the scalar field via $\phi(x^\alpha) \mapsto \varphi(x^\alpha) = \sqrt{2\omega_L + 3} \ln \frac{\phi}{\phi_0}$ or equivalently $\phi = \phi_0 \exp\left(\frac{\varphi}{\sqrt{2\omega_L + 3}}\right)$; then it follows that

$$\ln\Omega = \frac{1}{2}\ln\phi + \frac{1}{2}\ln G \\ = \frac{1}{2}\frac{\varphi}{\sqrt{2\omega_L + 3}} + \frac{1}{2}\ln\phi_0 + \frac{1}{2}\ln G. \quad (68)$$

With $\{\ln\phi_0, \ln G\}$ being constants, substitution of Eq.(68) into Eq.(67) yields

$$\sqrt{-g}\mathcal{G} = \sqrt{-\tilde{g}}\left(\tilde{\mathcal{G}} + \frac{\mathcal{K}(\tilde{\nabla}\varphi)}{\sqrt{2\omega_L + 3}}\right) \quad \text{and} \quad (69)$$

$$\mathcal{K}(\tilde{\nabla}\varphi) = -2\tilde{R}^{\alpha\beta}\left(2\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + \tilde{\nabla}_\alpha\varphi\tilde{\nabla}_\beta\varphi\right) \\ + \tilde{\Box}\varphi \cdot \left(2\tilde{R} + 2\tilde{\Box}\varphi - \tilde{\nabla}_\alpha\varphi\tilde{\nabla}^\alpha\varphi\right) \\ - 2\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi \cdot \left(\tilde{\nabla}^\alpha\tilde{\nabla}^\beta\varphi + \tilde{\nabla}^\alpha\varphi\tilde{\nabla}^\beta\varphi\right). \quad (70)$$

Here one can observe that since the coefficient Ω^{-4} in $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$ exactly neutralizes the Ω^4 in Eq.(67), the nonminimally ϕ -coupled Gauss-Bonnet effect $\int d^4x \sqrt{-g}\phi\mathcal{G}$ could never be canceled by a conformal rescaling $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$. Hence, the reduced LBD action Eq.(66) is finally transformed into

$$\mathcal{S} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - \frac{1}{2}\tilde{\nabla}_\alpha\varphi\tilde{\nabla}^\alpha\varphi + \right. \\ \left. b\phi_0 \exp\left(\frac{\varphi}{\sqrt{2\omega_L + 3}}\right) [\tilde{\mathcal{G}} + \mathcal{K}(\tilde{\nabla}\varphi)] \right\}, \quad (71)$$

which generalizes the canonical Gauss-Bonnet dark energy in vacuum $\mathcal{S} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{1}{2}\nabla_\alpha\varphi\nabla^\alpha\varphi + f(\varphi)\mathcal{G} \right)$ [12] by the Horndeski-like [30] or Galileon-like [31] kinetics in $\mathcal{K}(\tilde{\nabla}\varphi)$ for the scalar field.

Note that in the two examples just above, because of the nonminimal coupling to the scalar field $\phi(x^\alpha)$, negligible Gauss-Bonnet effect does not imply a zero Euler characteristic $\chi(\mathcal{M}) = \frac{1}{32\pi} \int \sqrt{-g}\mathcal{G}d^4x = 0$ for the spacetime, and similarly, negligibility of the Chern-Simons effect does not indicate a vanishing instanton number $\int {}^*RR d^4x = 0$, either.

Also, for the actions of the Chern-Simons gravity and the Gauss-Bonnet dark energy in the Jordan frame, in which a scalar field is respectively coupled to *RR and \mathcal{G} , we cannot

help but ask that why the scalar field is not simultaneously coupled to the Ricci scalar? We have previously seen from Eq.(47) that all algebraic and differential Riemannian invariants stand equal in front of the generalized Bianchi identities, so are there any good reasons for the scalar field to discriminate among different curvature invariants? We hope that the LBD gravity help release this tension (at least in empty space-times), as the scalar field ϕ indiscriminately couples to all the LBD invariants $\{R, {}^*RR, \mathcal{G}\}$, and the LBD gravity takes the Chern-Simons gravity and the Gauss-Bonnet dark energy as its reduced representations in the Einstein frame.

VII. COSMOLOGICAL APPLICATIONS

Having extensively discussed the theoretical structures of the LBD gravity, in this section we will apply this theory to the Friedman-Robertson-Walker (FRW) Universe and investigate the possibility to realize the late-time cosmic acceleration.

VII.1. Generalized Friedmann and Klein-Gordon equations

The field equation (35) can be recast into a GR form,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 G_{\text{eff}} \left(T_{\mu\nu}^{(\text{m})} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{CP})} + T_{\mu\nu}^{(\text{GB})} \right), \quad (72)$$

where $\kappa^2 = 8\pi$, and $G_{\text{eff}} = 1/\phi$ denotes the effective gravitational coupling strength. $T_{\mu\nu}^{(\text{m})} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{CP})} + T_{\mu\nu}^{(\text{GB})} =: T_{\mu\nu}^{(\text{eff})}$ comprises the total effective stress-energy-momentum tensor, with

$$\kappa^2 T_{\mu\nu}^{(\text{CP})} = -aH_{\mu\nu}^{(\text{CP})}, \quad \kappa^2 T_{\mu\nu}^{(\text{GB})} = -bH_{\mu\nu}^{(\text{GB})}, \quad \text{and} \quad (73)$$

$$\kappa^2 T_{\mu\nu}^{(\phi)} = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi + \frac{\omega_L}{\phi} \left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi \right).$$

Note that besides the effects of the source term $-\frac{\omega_L}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi$ in \mathcal{L}_{LBD} via $\delta\left(-\sqrt{-g}\frac{\omega_L}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi\right)/\delta g^{\mu\nu}$, the $(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi$ part from $\delta\left(\sqrt{-g}\phi R\right)/\delta g^{\mu\nu}$ is also packed into $T_{\mu\nu}^{(\phi)}$. Moreover, with the four distinct components of $T_{\mu\nu}^{(\text{eff})}$ sharing the same gravitational strength $1/\phi$, Eq.(72) implicitly respects the equivalence principle that the gravitational interaction is independent of the internal structures and compositions of a test body or self-gravitating object [21].

For the FRW metric of the flat Universe with a vanishing

spatial curvature index,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2, \quad (74)$$

* $RR = 0$ due to the maximal spatial symmetry, while the Ricci and Gauss-Bonnet scalars are respectively

$$\begin{aligned} R &= 6 \frac{a\ddot{a} + \dot{a}^2}{a^2} = 6(\dot{H} + 2H^2) \\ \mathcal{G} &= 24 \frac{\dot{a}^2 \ddot{a}}{a^3} = 24H^2(\dot{H} + H^2), \end{aligned} \quad (75)$$

where overdot denotes the derivative over the cosmic comoving time, and $H := \dot{a}/a$ represents the time-dependent Hubble parameter. Thus, an accelerated/decelerated flat Universe has a positive/negative Euler-Poincaré topological density. With a perfect-fluid form $T^\mu_\nu = \text{diag}[-\rho, P, P, P]$ assumed for each component in $T_{\mu\nu}^{(\text{eff})}$ [in consistency with the metric signature $(-, +, +, +)$], the cosmic expansion satisfies the generalized Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3\phi} \left(\kappa^2 \rho_m - 3H\dot{\phi} + \frac{\omega_L}{2\phi} \dot{\phi}^2 - 12bH^3\dot{\phi} \right), \quad (76)$$

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{1}{6\phi} \left\{ \kappa^2 (\rho_m + 3P_m) + 3\ddot{\phi} + 3H\dot{\phi} + \frac{2\omega_L}{\phi} \dot{\phi}^2 \right. \\ &\quad \left. + 12bH^2\ddot{\phi} + 12b(2\dot{H} + H^2)H\dot{\phi} \right\}, \end{aligned} \quad (77)$$

where $T_{\mu\nu}^{(\text{CP})} = 0$ for FRW. Moreover, the kinematical wave equation (37) and the dynamical wave equation (38) respectively lead to

$$\frac{2\omega_L}{\phi} (\ddot{\phi} + 3H\dot{\phi}) = 6 \frac{a\ddot{a} + \dot{a}^2}{a^2} + \frac{\omega_L}{\phi^2} + \ddot{\phi} + 24bH^2 \frac{\ddot{a}}{a}, \quad (78)$$

$$\begin{aligned} \left(2\omega_L + 3 + 12b \frac{a\ddot{a} + \dot{a}^2}{a^2} \right) (\ddot{\phi} + 3H\dot{\phi}) &= 24bH^2 \frac{\ddot{a}}{a} \phi \\ - 8\pi(3P_m - \rho_m) + 12b \left(\frac{\ddot{a}}{a} \ddot{\phi} + \frac{a\ddot{a} + 2\dot{a}^2}{a^2} H\dot{\phi} \right). \end{aligned} \quad (79)$$

In principle, one could understand the evolutions of the scale factor $a(t)$ and the homogeneous scalar field $\phi(t)$ by (probably numerically) solving Eqs.(76)-(79). However the solutions will be complicated, so we will start with some solution ansatz for $\{a(t), \phi(t)\}$, which are easier to work with.

VII.2. Cosmic acceleration in the late-time approximation

The physical matter satisfies the continuity equation

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0, \quad (80)$$

and for pressureless dust $P_m = 0$, it integrates to yield

$$\rho_m = \rho_0^{(m)} a^{-3} = \frac{\rho_0^{(m)}}{a_0^3} t^{-3\beta}, \quad (81)$$

where we have assumed a power-law scale factor

$$a = a_0 t^\beta \quad \text{with } \beta > 1. \quad (82)$$

Here $\{a_0, \beta\}$ are constants, and $\beta > 1$ so that $\ddot{a} > 0$. Similarly, we also take a power-law ansatz for the scalar field,

$$\phi = \phi_0 t^\gamma. \quad (83)$$

Based on Eqs.(81)-(83), the dynamical wave equation (38) with $T^{(m)} = -\rho_m$ for dust yields

$$\begin{aligned} \gamma(2\omega_L + 3)(3\beta - 1 + \gamma) &= \frac{\kappa^2 \rho_0^{(m)}}{\phi_0 a_0^3} t^{2-3\beta-\gamma} \\ &\quad + 24b \frac{\beta^3(\beta - 1)}{t^2} - 12b\beta^2 \gamma \frac{(3\beta - 3 + \gamma)}{t^2}, \end{aligned} \quad (84)$$

and in the late-time (large t) approximation it reduces to

$$\gamma(\gamma + 3\beta - 1) = \frac{1}{\phi_0 a_0^3} \frac{\kappa^2 \rho_0^{(m)}}{(2\omega_L + 3)} t^{2-3\beta-\gamma}, \quad (85)$$

which can be satisfied by

$$\gamma = 2 - 3\beta \quad \text{and} \quad \phi_0 = \frac{\kappa^2 \rho_0^{(m)}}{a_0^3 (2\omega_L + 3) (2 - 3\beta)}. \quad (86)$$

Moreover, the first Friedmann equation (76) leads to

$$3\beta^2 = \kappa^2 \frac{\rho_0^{(m)}}{a_0^3 \phi_0} t^{2-3\beta-\gamma} - 3\beta\gamma + \frac{\omega_L}{2} \gamma^2 - 12b \frac{\beta^3 \gamma}{t^2}, \quad (87)$$

and with Eq.(86), in the late-time approximation it becomes

$$3\beta^2 = (2 - 3\beta)(2\omega_L + 3) - 3\beta(2 - 3\beta) + \frac{\omega_L}{2} (2 - 3\beta)^2. \quad (88)$$

For $\beta = 2$, Eq.(88) trivially holds for an arbitrary ω_L , while for $\beta \neq 2$, we have β in terms of ω_L via

$$\beta = \frac{2(\omega_L + 1)}{3\omega_L + 4}. \quad (89)$$

Note that Eqs.(86) and (89) require $\omega_L \neq -4/3$, $\omega_L \neq -3/2$ ($\beta \neq 2$), and $\beta \neq 2/3$; they are simply consequences of the power-law-solution ansatz and the late-time approximations rather than universal constraints on ω_L , and according to Eq.(89), the last condition $\beta \neq 2/3$ trivially holds with $\beta \rightarrow 2/3$ for $\omega_L \rightarrow \infty$. As a consistency test, the kinematical equation (37) yields

$$\omega_L \left(\frac{1}{2} \gamma^2 - \gamma + 3\beta\gamma \right) = 3\beta(2\beta - 1) + 12b \frac{\beta^3(\beta - 1)}{t^2} \quad (90)$$

with the late-time approximation

$$\omega_L \left(\frac{1}{2} \gamma^2 - \gamma + 3\beta\gamma \right) = 3\beta(2\beta - 1), \quad (91)$$

which holds for Eqs.(86) and (89). Substituting Eqs.(81), (82), (83), (86) and (89) into the second Friedmann equation (77), we obtain

$$\frac{\ddot{a}}{a} = -\frac{2(\omega_L + 1)(\omega_L + 2)}{(3\omega_L + 4)^2} t^{-2}, \quad (92)$$

and the deceleration parameter reads

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2} \left(1 + 3 \frac{P_{\text{eff}}}{\rho_{\text{eff}}} \right) = \frac{\omega_L + 2}{2(\omega_L + 1)}. \quad (93)$$

Eqs.(92) and (93) clearly indicate that the late-time acceleration could be realized for $-2 < \omega_L < -1$ ($\omega_L \neq -4/3$, $\omega_L \neq -3/2$), although this domain of ω_L makes the kinetics of the scalar field noncanonical.

VIII. LOVELOCK-SCALAR-TENSOR GRAVITY

VIII.1. From LBD to Lovelock-scalar-tensor gravity

The LBD gravity can be generalized into the Lovelock-scalar-tensor (LST) gravity with the action

$$\begin{aligned} \mathcal{S}_{\text{LST}} &= \int d^4x \sqrt{-g} \mathcal{L}_{\text{LST}} + \mathcal{S}_m \quad \text{and} \\ \mathcal{L}_{\text{LST}} &= \frac{1}{16\pi G} \left(f_1(\phi)R + f_2(\phi) \frac{^*RR}{\sqrt{-g}} + f_3(\phi)\mathcal{G} \right. \\ &\quad \left. - \frac{\omega(\phi)}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) \right), \end{aligned} \quad (94)$$

where $\{f_i(\phi), \omega(\phi)\}$ are generic functions of the scalar field, and $V(\phi)$ is the self-interaction potential. Note that this time Newton's constant G is included in the overall coefficient $1/16\pi G$ of \mathcal{L}_{LST} , as is the case of the ordinary scalar-tensor gravity. The gravitational field equation is

$$\begin{aligned} f_1(\phi) \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) &+ (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) f_1(\phi) \\ &- \frac{\omega(\phi)}{\phi} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) \\ &+ \frac{1}{2} V(\phi) g_{\mu\nu} + \tilde{H}_{\mu\nu}^{(\text{CP})} + \tilde{H}_{\mu\nu}^{(\text{GB})} = 8\pi T_{\mu\nu}^{(\text{m})}, \end{aligned} \quad (95)$$

where $\tilde{H}_{\mu\nu}^{(\text{CP})}$ denotes the contribution from $f_2(\phi)^*RR$,

$$\begin{aligned} \sqrt{-g} \tilde{H}_{\mu\nu}^{(\text{CP})} &= 2\partial^\epsilon f_2(\phi) \cdot (\epsilon_{\xi\mu\alpha\beta} \nabla^\alpha R^\beta{}_\nu + \epsilon_{\xi\nu\alpha\beta} \nabla^\alpha R^\beta{}_\mu) \\ &+ 2\partial_\alpha \partial_\beta f_2(\phi) \cdot (^*R^\alpha{}_\mu{}^\beta{}_\nu + ^*R^\alpha{}_\nu{}^\beta{}_\mu), \end{aligned} \quad (96)$$

and $\tilde{H}_{\mu\nu}^{(\text{GB})}$ attributes to the effect of $\sqrt{-g}f_3(\phi)\mathcal{G}$,

$$\begin{aligned} \tilde{H}_{\mu\nu}^{(\text{GB})} &= 2R(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) f_3(\phi) - 4R_{\mu\nu}\square f_3(\phi) + \\ &4R_\mu{}^\alpha \nabla_\alpha \nabla_\nu f_3(\phi) + 4R_\nu{}^\alpha \nabla_\alpha \nabla_\mu f_3(\phi) - \\ &4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta f_3(\phi) + 4R_{\alpha\mu\beta\nu} \nabla^\beta \nabla^\alpha f_3(\phi). \end{aligned} \quad (97)$$

It is straightforward to derive the kinematical wave equation by $\delta\mathcal{S}_{\text{LST}}/\delta\phi = 0$, which along with the trace of Eq.(95) could yield the dynamical wave equation, and they generalize the wave equations (37, 38) in the LBD gravity. The wave equations however will not be listed here as the interest of this section is only the field equation $\delta\mathcal{S}/\delta g^{\mu\nu} = 0$.

VIII.2. Equivalence of LST with fourth-order gravities

It is well known that the $f(R)$ gravity is equivalent to the nondynamical (i.e. $\omega_{\text{BD}} = 0$) Brans-Dicke gravity [27], and such equivalence holds for the LBD gravity as well. Consider the fourth-order modified gravity

$$\mathcal{L} = \frac{1}{16\pi G} \left[f(R, \mathcal{G}) + h \left(\frac{^*RR}{\sqrt{-g}} \right) \right], \quad (98)$$

for which the field equation is

$$\begin{aligned} f_R R_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) f_R - \frac{1}{2} f(R, \mathcal{G}) g_{\mu\nu} \\ + \mathcal{H}_{\mu\nu}^{(\text{CP})} + \mathcal{H}_{\mu\nu}^{(\text{GB})} = 8\pi T_{\mu\nu}^{(\text{m})}, \end{aligned} \quad (99)$$

where

$$\begin{aligned} \sqrt{-g} \mathcal{H}_{\mu\nu}^{(\text{CP})} &= 2\partial^\epsilon h_{^*RR} \cdot (\epsilon_{\xi\mu\alpha\beta} \nabla^\alpha R^\beta{}_\nu + \epsilon_{\xi\nu\alpha\beta} \nabla^\alpha R^\beta{}_\mu) \\ &+ 2\partial_\alpha \partial_\beta h_{^*RR} \cdot (^*R^\alpha{}_\mu{}^\beta{}_\nu + ^*R^\alpha{}_\nu{}^\beta{}_\mu), \end{aligned} \quad (100)$$

and

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(\text{GB})} &= 2R(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) f_{\mathcal{G}} - 4R_{\mu\nu}\square f_{\mathcal{G}} + \\ &4R_\mu{}^\alpha \nabla_\alpha \nabla_\nu f_{\mathcal{G}} + 4R_\nu{}^\alpha \nabla_\alpha \nabla_\mu f_{\mathcal{G}} - \\ &4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta f_{\mathcal{G}} + 4R_{\alpha\mu\beta\nu} \nabla^\beta \nabla^\alpha f_{\mathcal{G}}, \end{aligned} \quad (101)$$

with $f_R = f_R(R, \mathcal{G}) = \partial f(R, \mathcal{G})/\partial R$, $f_{\mathcal{G}} = f_{\mathcal{G}}(R, \mathcal{G}) = \partial f(R, \mathcal{G})/\partial \mathcal{G}$, and $h_{^*RR} = dh(^*RR/\sqrt{-g})/d(^*RR/\sqrt{-g})$. For the nondynamical LST gravity with $\omega(\phi) \equiv 0$ in Eq.(95), compare it with Eq.(99) and at the level of the gravitational equation, one could find the equivalence

$$\begin{aligned} f_1(\phi) &= f_R, \quad f_3(\phi) = f_{\mathcal{G}}, \quad f_2(\phi) = h_{^*RR}, \\ V(\phi) &= -f(R, \mathcal{G}) + f_R R. \end{aligned} \quad (102)$$

In the $V(\phi)$ relation we have applied the replacement $f_1(\phi) = f_R$, and note that $V(\phi)$ does not contain a $f_{\mathcal{G}}\mathcal{G}$ term which has been removed from $\mathcal{H}_{\mu\nu}^{(\text{GB})}$ because of the Bach-Lanczos identity Eq.(32).

VIII.3. Partial equivalence for “multi-scalar LBD gravity”

Removing the ω_L term in Eq.(35) and then comparing it with Eq.(99), one could find that an equivalence between the nondynamical LBD gravity (now equipped with an extra potential $-U(\phi)$ in \mathcal{L}_{LBD}) and the $f(R, \mathcal{G}) + h\left(\frac{*RR}{\sqrt{-g}}\right)$ gravity would require $f_R = f_{\mathcal{G}} = \phi = h_{*RR}$, and $U(\phi) = -f(R, \mathcal{G}) + f_R R$. These conditions are so restrictive that the $f(R, \mathcal{G}) + h\left(\frac{*RR}{\sqrt{-g}}\right)$ gravity would totally lose its generality. Instead, introduce three auxiliary fields $\{\chi_1, \chi_2, \chi_3\}$ and consider the dynamically equivalent action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[f(\chi_1, \chi_2) + f_{\chi_1} \cdot (R - \chi_1) + f_{\chi_2} \cdot (\mathcal{G} - \chi_2) + h(\chi_3) + h_{\chi_3} \cdot \left(\frac{*RR}{\sqrt{-g}} - \chi_3 \right) \right] + S_m; \quad (103)$$

its variation with respect to χ_1 , χ_2 , and χ_3 separately yields the constraints

$$f_{\chi_1 \chi_1} (R - \chi_1) = 0, \quad f_{\chi_2 \chi_2} (\mathcal{G} - \chi_2) = 0, \\ \text{and} \quad h_{\chi_3 \chi_3} \left(\frac{*RR}{\sqrt{-g}} - \chi_3 \right) = 0, \quad (104)$$

where $f_{\chi_j} := \partial f(\chi_1, \chi_2) / \partial \chi_j$, $f_{\chi_j \chi_j} := \partial^2 f(\chi_1, \chi_2) / \partial \chi_j^2$, $h_{\chi_3} := \partial h(\chi_3) / \partial \chi_3$ and $h_{\chi_3 \chi_3} := \partial^2 h(\chi_3) / \partial \chi_3^2$. If $f_{\chi_1 \chi_1}$, $f_{\chi_2 \chi_2}$ and $h_{\chi_3 \chi_3}$ do not vanish identically, Eq.(104) leads to $\chi_1 = R$, $\chi_2 = \mathcal{G}$ and $\chi_3 = \frac{*RR}{\sqrt{-g}}$. Redefining the fields $\{\chi_1, \chi_2, \chi_3\}$ by

$$\phi = f_{\chi_1}, \quad \psi = f_{\chi_2}, \quad \varphi = h_{\chi_3} \quad (105)$$

and setting

$$V(\phi, \psi, \varphi) = \phi \cdot R(\phi, \psi) + \psi \cdot \mathcal{G}(\phi, \psi) + \varphi \cdot \frac{*RR}{\sqrt{-g}}(\varphi) - f(R(\phi, \psi), \mathcal{G}(\phi, \psi)) - h\left(\frac{*RR}{\sqrt{-g}}(\varphi)\right), \quad (106)$$

then the $f(R, \mathcal{G}) + h\left(\frac{*RR}{\sqrt{-g}}\right)$ gravity is partially equivalent to the following “multi-scalar LBD gravity” carrying three nondynamical scalar fields

$$\mathcal{L} = \frac{1}{16\pi} \left(\phi R + \varphi \frac{*RR}{\sqrt{-g}} + \psi \mathcal{G} - V(\phi, \psi, \varphi) \right), \quad (107)$$

where the coupling coefficients $\{a, b\}$ appearing in \mathcal{L}_{LBD} have been absorbed into the scalar fields $\{\varphi, \psi\}$. Also, by “partially equivalent” we mean that Eq.(106) as it stands is only partially on-shell; to recover Eq.(98) from the multi-field action of Eq.(107), one would have to add extra Lagrange multipliers identifying the different fields, but this would break the exact equivalence between such modified Eq.(107) and Eq.(98).

IX. CONCLUSIONS AND DISCUSSION

The Hilbert-Einstein action S_{HE} and the Lovelock action S_L yield identical field equations and thus are observationally indistinguishable. However, the former takes the Brans-Dicke gravity as its scalar-tensor counterpart, while the latter’s companion is the LBD gravity, and these two theories are different.

We have extensively studied the theoretical structures of the LBD gravity, including the gravitational and wave equations, the ordinary $\omega_L \rightarrow \infty$ limit that recovers GR, the unusual $\omega_L \rightarrow \infty$ limit satisfying the topology balance condition Eq.(41) and thus departing from GR, the energy-momentum conservation, the conformal transformations into the dynamical Chern-Simons gravity and the generalized Gauss-Bonnet dark energy, as well as the extensions to LST gravity with its equivalence to fourth-order modified gravity.

We have taken the opportunity of deriving the field equation to look deeper into the properties of the Chern-Pontryagin and Gauss-Bonnet topological invariants. Especially, for the $f(\phi)\mathcal{G}$ Gauss-Bonnet dark energy as well as the $f(R, \mathcal{G})$ and $f(R, \mathcal{G}, \mathcal{L}_m)$ gravities, the contributions of the generalized Gauss-Bonnet dependence could be simplified from the popular form like Eq.(31) into our form like Eq.(33).

An important goal of alternative and modified gravities is to explain the accelerated expansion of the Universe, and we have applied the LBD theory to this problem, too. It turned out that the acceleration could be realized for $-2 < \omega_L < -1$ under our solution ansatz. Note that our estimate of cosmic acceleration in Sec. VII is not satisfactory. For example, the kinematical equation (90) clearly shows that because of the higher-order time derivative terms arising from the $\phi\mathcal{G}$ dependence, the simplest solution ansatz $\{\phi = \phi_0 t^\gamma, a = a_0 t^\beta\}$ with $\{\beta = \text{constant}, \gamma = \text{constant}\}$ are not compatible with each other unless the late-time approximation is imposed, while such approximations further lead to the behaviors analogous to the Brans-Dicke cosmology [32].

Section VII has shown that, the effects from the parity-violating Chern-Pontryagin term ϕ^*RR are ineffective for the FRW cosmology because of its spatial homogeneity and isotropy. However, it is believed that ϕ^*RR could have detectable consequences on leptogenesis and gravitational waves in the initial inflation epoch [33] where ϕ acts as the inflaton field. The inflation problem usually works with the slow-roll approximations $\ddot{\phi} \ll \dot{\phi} \ll H$ and requires the existence of a potential well $V(\phi)$; thus, at least for the description of the initial inflation, the LBD gravity should be generalized to carry a potential:

$$\widehat{\mathcal{L}}_{\text{LBD}} = \frac{1}{16\pi} \left[\phi \left(R + a \frac{*RR}{\sqrt{-g}} + b\mathcal{G} \right) - \frac{\omega_L}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) \right], \quad (108)$$

with $V(\phi) = 2\Lambda\phi$ being the simplest possibility.

Our prospective studies aim to construct the complete history of cosmic expansion in LBD gravity [probably equipped with $V(\phi)$], throughout the dominance of radiation, dust, and effective dark energy. Moreover, it is well known that primordial gravitational waves can trace back to the Planck era of the Universe and serve as one of the most practical and effi-

cient tests for modified gravities, so it is very useful to find out whether the gravitational-wave polarizations carry different intensities in this gravity. There are also some other problems from the LBD gravity attracting our attention, such as its relation to the low-energy effective string theory. We will look for the answers in future.

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- [24] Switching from the sign convention in Ref.[11] to the convention in our paper and Ref.[20], the signs for the following quantities are changed: $g_{\mu\nu}$, $\Gamma_{\alpha\beta\gamma}$, $R^\alpha{}_{\mu\beta\nu}$, $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, $T_{\mu\nu}^{(m)}$, Λ (cosmological constant) while the following quantities take unchanging values: $\sqrt{-g}$, $\epsilon^{\alpha\beta\mu\nu}$, $\epsilon_{\alpha\beta\mu\nu}$, $\Gamma_{\mu\nu}^\alpha$, $R_{\alpha\mu\beta\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$. Moreover, Ref.[11] defined *RR by $(\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta})R^{\alpha\beta\gamma\delta} = 2{}^*R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = -2{}^*RR$, differing from ours by a minus sign, as we choose to follow Lovelock's usage which is more popular and standard.
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